THE 2-CATEGORY OF WEAK ENTWINING STRUCTURES

GABRIELLA BÖHM

Abstract. A weak entwining structure in a 2-category $\mathcal{K}$ consists of a monad t and a comonad c, together with a 2-cell relating both structures in a way which generalizes a mixed distributive law. A weak entwining structure can be characterized as a compatible pair of a monad and a comonad, in 2-categories generalizing the 2-category of comonads and the 2-category of monads in $\mathcal{K}$, respectively. This observation is used to define a 2-category $\mathrm{Entw}^w(\mathcal{K})$ of weak entwining structures in $\mathcal{K}$. If the 2-category $\mathcal{K}$ admits Eilenberg-Moore constructions for both monads and comonads and idempotent 2-cells in $\mathcal{K}$ split, then there are pseudo-functors from $\mathrm{Entw}^w(\mathcal{K})$ to the 2-category of monads and to the 2-category of comonads in $\mathcal{K}$, taking a weak entwining structure (t, c) to a 'weak lifting' of t for c and a 'weak lifting' of c for t, respectively. The Eilenberg-Moore objects of the lifted monad and the lifted comonad are shown to be equivalent. If $\mathcal{K}$ is the 2-category of functors induced by bimodules, then these Eilenberg-Moore objects are isomorphic to the usual category of weak entwined modules.

Introduction

Mixed distributive laws [1] in a 2-category $\mathcal{K}$ (or 'entwining structures', as they are called more often in the Hopf algebraic terminology), can be described in some equivalent ways [8]. They are monads in the 2-category $\mathrm{Cmd}(\mathcal{K})$ of comonads in $\mathcal{K}$, equivalently, they are comonads in the 2-category $\mathrm{Mnd}(\mathcal{K})$ of monads in $\mathcal{K}$. Consequently, they can be regarded as 0-cells of a 2-category $\mathrm{Entw}(\mathcal{K})$, defined to be isomorphic to $\mathrm{Mnd}(\mathrm{Cmd}(\mathcal{K})) \cong \mathrm{Cmd}(\mathrm{Mnd}(\mathcal{K}))$.

If a 2-category $\mathcal{K}$ admits Eilenberg-Moore constructions for monads, that is, the inclusion 2-functor $I : \mathcal{K} \to \mathrm{Mnd}(\mathcal{K})$ possesses a right 2-adjoint J, then the 2-functor $\mathrm{Cmd}(J)$ takes a mixed distributive law of a monad t and a comonad c in $\mathcal{K}$ to a comonad $J(t) \xrightarrow{\overline{c}} J(t)$, which is a lifting of c, cf. [7]. Symmetrically, if $\mathcal{K}$ admits Eilenberg-Moore constructions for comonads, that is, the inclusion 2-functor $I_* : \mathcal{K} \to \mathrm{Cmd}(\mathcal{K})$ possesses a right 2-adjoint J_*, then $\mathrm{Mnd}(J_*)$ takes (t, c) to a monad $J_*(c) \xrightarrow{\overline{t}} J_*(c)$, which is a lifting of t. If Eilenberg-Moore constructions in $\mathcal{K}$ exist both for monads and comonads, then the 2-functors $J_*\mathrm{Cmd}(J)$ and $J\mathrm{Mnd}(J_*)$ are 2-naturally isomorphic. In particular, the lifted monad $\overline{t}$ and the lifted comonad $\overline{c}$ possess isomorphic Eilenberg-Moore objects, see [7]. In the case when $\mathcal{K}$ is the 2-category CAT = `[Categories; Functors; Natural Transformations]`, this is the category of (t, c)-bimodules, also called 'entwined modules'.

In order to treat algebra extensions by weak bialgebras in [3], entwining structures were generalized to 'weak entwining structures' in [5]. A weak entwining structure in a 2-category $\mathcal{K}$ also consists of a monad t and a comonad c, together with a 2-cell $tc \Rightarrow ct$, but the compatibility axioms with the unit of the monad and the counit of the comonad are weakened. We are not aware of any characterization of a weak

entwining structure as a monad or as a comonad in some 2-category. Instead, in this note we observe that a weak entwining structure in an arbitrary 2-category $\mathcal{K}$ can be described as a compatible pair of a comonad in a 2-category $\mathrm{Mnd}^\iota(\mathcal{K})$, which extends $\mathrm{Mnd}(\mathcal{K})$, and a monad in $\mathrm{Cmd}^\pi(\mathcal{K}) := \mathrm{Mnd}^\iota(\mathcal{K}_*)_*$ (where $(-)_*$ means the vertically opposite 2-category). This observation is used to define in Section 1 a 2-category $\mathrm{Entw}^w(\mathcal{K})$, whose 0-cells are weak entwining structures in $\mathcal{K}$ and whose 1-cells and 2-cells are also compatible pairs of 1-cells and 2-cells, respectively, in $\mathrm{Mnd}(\mathrm{Cmd}^\pi(\mathcal{K}))$ and $\mathrm{Cmd}(\mathrm{Mnd}^\iota(\mathcal{K}))$. By construction, the 2-category $\mathrm{Entw}^w(\mathcal{K})$ comes equipped with 2-functors $A : \mathrm{Entw}^w(\mathcal{K}) \to \mathrm{Cmd}(\mathrm{Mnd}^\iota(\mathcal{K}))$ and $B : \mathrm{Entw}^w(\mathcal{K}) \to \mathrm{Mnd}(\mathrm{Cmd}^\pi(\mathcal{K}))$.

If a 2-category $\mathcal{K}$ admits Eilenberg-Moore constructions for monads and idempotent 2-cells in $\mathcal{K}$ split, then the 2-functor J above factorizes through the inclusion $\mathrm{Mnd}(\mathcal{K}) \hookrightarrow \mathrm{Mnd}^\iota(\mathcal{K})$ and an appropriate pseudo-functor $Q : \mathrm{Mnd}^\iota(\mathcal{K}) \to \mathcal{K}$. The image of a weak entwining structure (t, c) under the pseudo-functor $\mathrm{Cmd}(Q)A$ is a 'weak lifting' of c for t, cf. [2]. Symmetrically, if $\mathcal{K}$ admits Eilenberg-Moore constructions for comonads and idempotent 2-cells in $\mathcal{K}$ split, then there is a pseudo-functor $Q_* : \mathrm{Cmd}^\pi(\mathcal{K}) \to \mathcal{K}$, such that $\mathrm{Mnd}(Q_*)B$ takes a weak entwining structure (t, c) to a weak lifting of t for c. If Eilenberg-Moore constructions in $\mathcal{K}$ exist both for monads and comonads and also idempotent 2-cells in $\mathcal{K}$ split, then we prove in Section 2 that the pseudo-functors $J_*\mathrm{Cmd}(Q)A$ and $J\mathrm{Mnd}(Q_*)B : \mathrm{Entw}^w(\mathcal{K}) \to \mathcal{K}$ are pseudo-naturally equivalent. In particular, for any weak entwining structure (t, c), the weak lifting of t for c, and the weak lifting of c for t, possess equivalent Eilenberg-Moore objects.

As a motivating example, we can consider the 2-category $\mathcal{K}$ obtained as the image of the bicategory $\mathrm{BIM}_k =$`[Algebras; Bimodules; Bimodule Maps]` (over a commutative ring k) under the hom 2-functor $\mathrm{BIM}_k(k, -) : \mathrm{BIM}_k \to \mathrm{CAT}$. A weak entwining structure $((-)\otimes_R T, (-)\otimes_R C)$ in this 2-category is given by a k-algebra R, an R-ring T, an R-coring C and an R-bimodule map $C \otimes_R T \to T \otimes_R C$. In this case, we obtain that the Eilenberg-Moore category of the weakly lifted comonad $\overline{(-) \otimes_R C}$ (on the category M_T of T-modules) is isomorphic to the Eilenberg-Moore category of the weakly lifted monad $\overline{(-) \otimes_R T}$ (on the category M^C of C-comodules), and it is isomorphic also to $\mathrm{Entw}^w(\mathcal{K})((M_k, M_k), ((-)\otimes_R T, (-)\otimes_R C))$, known as the category of 'weak entwined modules'. In particular, if R is a trivial k-algebra (i.e. $R = k$), we re-obtain [4, Proposition 2.3].

Notations. We assume that the reader is familiar with the theory of 2-categories. For a review of the occurring notions (such as a 2-category, a 2-functor and a 2-adjunction, monads, adjunctions and Eilenberg-Moore construction in a 2-category) we refer to the article [6].

In a 2-category $\mathcal{K}$, horizontal composition is denoted by juxtaposition and vertical composition is denoted by $*$, 1-cells are represented by an arrow $\to$ and 2-cells are represented by $\Rightarrow$.

For any 2-category $\mathcal{K}$, $\mathrm{Mnd}(\mathcal{K})$ denotes the 2-category of monads in $\mathcal{K}$ as in [8] and $\mathrm{Cmd}(\mathcal{K}) := \mathrm{Mnd}(\mathcal{K}_*)_*$ denotes the 2-category of comonads in $\mathcal{K}$, where $(-)_*$ refers to the vertical opposite of a 2-category. Throughout, we denote by $I : \mathcal{K} \to \mathrm{Mnd}(\mathcal{K})$ the inclusion 2-functor (with underlying maps $k \mapsto (k, k, k)$, $V \mapsto (V, V)$, $\omega \mapsto \omega$ on the 0-, 1-, and 2-cells, respectively). Its right 2-adjoint, if it exists, is denoted by J. The

inclusion 2-functor $\mathcal{K} \to \mathrm{Cmd}(\mathcal{K})$ is denoted by I_* and its right 2-adjoint, whenever it exists, is denoted by J_*.

If a 2-category $\mathcal{K}$ admits Eilenberg-Moore constructions for monads (i.e. the 2-functor J exists), then any monad $(k \xrightarrow{t} k, tt \overset{\mu}{\Rightarrow} t, k \overset{\eta}{\Rightarrow} t)$ in $\mathcal{K}$ determines a canonical adjunction $(k \xrightarrow{f} J(t), J(t) \xrightarrow{v} k, fv \overset{\epsilon}{\Rightarrow} J(t), k \overset{\eta}{\Rightarrow} vf)$ such that $(t, \mu, \eta) = (vf, v\epsilon f, \eta)$, cf. [8, Theorem 2]. Throughout, these notations are used for this canonical adjunction. For a monad (t', μ', η'), the canonical adjunction is denoted by $(f', v', \epsilon', \eta')$, etc.

We say that in a 2-category $\mathcal{K}$ idempotent 2-cells split if, for any 2-cell $V \overset{\Theta}{\Rightarrow} V$ in $\mathcal{K}$ such that $\Theta * \Theta = \Theta$, there exist a 1-cell $\widehat{V}$ and 2-cells $V \overset{\pi}{\Rightarrow} \widehat{V}$ and $\widehat{V} \overset{\iota}{\Rightarrow} V$, such that $\pi * \iota = \widehat{V}$ and $\iota * \pi = \Theta$.

1. The 2-category of weak entwining structures

Consider a monad $(k \xrightarrow{t} k, tt \overset{\mu}{\Rightarrow} t, k \overset{\eta}{\Rightarrow} t)$ and a comonad $(k \xrightarrow{c} k, c \overset{\delta}{\Rightarrow} cc, c \overset{\varepsilon}{\Rightarrow} k)$ in a 2-category $\mathcal{K}$ and a 2-cell $tc \overset{\psi}{\Rightarrow} ct$. The triple (t, c, ψ) is termed a *weak entwining structure* provided that the following axioms in [5] hold.

$$\psi * \mu c = c\mu * \psi t * t\psi; \tag{1.1}$$

$$\delta t * \psi = c\psi * \psi c * t\delta; \tag{1.2}$$

$$\psi * \eta c = c\varepsilon t * c\psi * c\eta c * \delta; \tag{1.3}$$

$$\varepsilon t * \psi = \mu * t\varepsilon t * t\psi * t\eta c. \tag{1.4}$$

The most important difference between such a weak entwining structure and a usual entwining structure (i.e. mixed distributive law) is that in the weak case (c, ψ) is no longer a 1-cell $t \to t$ in $\mathrm{Mnd}(\mathcal{K})$ and (t, ψ) is not a 1-cell $c \to c$ in $\mathrm{Cmd}(\mathcal{K})$. Still, as it was observed in [2], $(t \xrightarrow{(c,\psi)} t, \mu, \eta)$ is a monad and $(c \xrightarrow{(t,\psi)} c, \delta, \varepsilon)$ is a comonad in an extended 2-category of (co)monads in $\mathcal{K}$, recalled in the following theorem.

Theorem 1.1 ([2], Corollary 1.4 and Theorem 3.5)**.** *For any 2-category $\mathcal{K}$, the following data constitute a 2-category, to be denoted by $\mathrm{Mnd}^\iota(\mathcal{K})$.*

0-cells are monads $(k \xrightarrow{t} k, \mu, \eta)$ in $\mathcal{K}$.
1-cells $(k \xrightarrow{t} k, \mu, \eta) \xrightarrow{(V,\psi)} (k' \xrightarrow{t'} k', \mu', \eta')$ are pairs, consisting of a 1-cell $k \xrightarrow{V} k'$ and a 2-cell $t'V \overset{\psi}{\Rightarrow} Vt$ in $\mathcal{K}$ such that

$$V\mu * \psi t * t'\psi = \psi * \mu' V. \tag{1.5}$$

2-cells $(V, \psi) \overset{\omega}{\Rightarrow} (W, \phi)$ are 2-cells $V \overset{\omega}{\Rightarrow} W$ in $\mathcal{K}$, satisfying

$$\omega t * \psi = W\mu * \phi t * t'\omega t * t'\psi * t'\eta' V. \tag{1.6}$$

Horizontal and vertical compositions are the same as in $\mathcal{K}$.

The 2-category $\mathrm{Mnd}^\iota(\mathcal{K})$ contains $\mathrm{Mnd}(\mathcal{K})$ as a vertically full 2-subcategory.

Moreover, if $\mathcal{K}$ admits Eilenberg-Moore constructions for monads and idempotent 2-cells in $\mathcal{K}$ split, then the following maps determine a pseudo-functor $Q : \mathrm{Mnd}^\iota(\mathcal{K}) \to \mathcal{K}$.

For a 0-cell (t, μ, η), $Q(t, \mu, \eta) := J(t, \mu, \eta)$.

For a <u>*1-cell*</u> $(t,\mu,\eta) \stackrel{(V,\psi)}{\to} (t',\mu',\eta')$, $Q(V,\psi)$ *is the unique 1-cell* $Q(t,\mu,\eta) \to Q(t',\mu',\eta')$ *in* $\mathcal{K}$ *for which*

$$v'\epsilon' Q(V,\psi) = \pi * Vv\epsilon * \psi v * t'\iota. \tag{1.7}$$

For a <u>*2-cell*</u> $(V,\psi) \stackrel{\omega}{\Rightarrow} (W,\phi)$, $Q(\omega)$ *is the unique 2-cell* $Q(V,\psi) \Rightarrow Q(W,\phi)$ *in* $\mathcal{K}$ *for which*

$$v'Q(\omega) = \pi * \omega v * \iota, \tag{1.8}$$

where $Vv \stackrel{\pi}{\Rightarrow} v'Q(V,\psi) \stackrel{\iota}{\Rightarrow} Vv$ *denote a chosen splitting of the idempotent 2-cell*

$$Vv\epsilon * \psi v * \eta' Vv : Vv \Rightarrow Vv, \tag{1.9}$$

for any 1-cell (V,ψ) *in* $\mathrm{Mnd}^\iota(\mathcal{K})$.

For 1-cells $t \stackrel{(V,\psi)}{\to} t' \stackrel{(V',\psi')}{\to} t''$ *in* $\mathrm{Mnd}^\iota(\mathcal{K})$, *the coherence natural iso 2-cell* $Q((V',\psi')(V,\psi)) \stackrel{\cong}{\Rightarrow} Q(V',\psi')Q(V,\psi)$ *is the unique 2-cell* γ *for that* $v''\gamma = \big(v''Q((V',\psi')(V,\psi)) \stackrel{\iota}{\Rightarrow} V'Vv \stackrel{V'\pi}{\Rightarrow} V'v'Q(V,\psi) \stackrel{\pi Q(V,\psi)}{\Rightarrow} v''Q(V',\psi')Q(V,\psi)\big)$ *(so* $v''\gamma^{-1} = \big(v''Q(V',\psi')Q(V,\psi) \stackrel{\iota Q(V,\psi)}{\Rightarrow} V'v'Q(V,\psi) \stackrel{V'\iota}{\Rightarrow} V'Vv \stackrel{\pi}{\Rightarrow} v''Q((V',\psi')(V,\psi))\big)$*). With the convention of choosing a trivial splitting* $Vv \stackrel{Vv}{\Rightarrow} Vv \stackrel{Vv}{\Rightarrow} Vv$ *whenever* (1.9) *is an identity 2-cell, the image of any identity 1-cell* $t \stackrel{(k,t)}{\to} t$ *under* Q *becomes equal to the identity 1-cell* $Q(t)$. *This convention also ensures that the composite pseudo-functor* $\mathrm{Mnd}(\mathcal{K}) \hookrightarrow \mathrm{Mnd}^\iota(\mathcal{K}) \stackrel{Q}{\to} \mathcal{K}$ *is equal to* J. *The pseudo-natural isomorphism class of* Q *does not depend on the choice of the 2-cells* π *and* ι.

For any 2-category $\mathcal{K}$, we put $\mathrm{Cmd}^\pi(\mathcal{K}) := \mathrm{Mnd}^\iota(\mathcal{K}_*)_*$. Applying Theorem 1.1 to the 2-category $\mathcal{K}_*$, we conclude that whenever $\mathcal{K}$ admits Eilenberg-Moore constructions for comonads and idempotent 2-cells in $\mathcal{K}$ split, J_* extends to a pseudo-functor $Q_* : \mathrm{Cmd}^\pi(\mathcal{K}) \to \mathcal{K}$.

After all these preparations, we are ready to construct a 2-category of weak entwining structures in any 2-category $\mathcal{K}$.

Theorem 1.2. *For any 2-category* $\mathcal{K}$, *the following data constitute a 2-category, to be denoted by* $\mathrm{Entw}^w(\mathcal{K})$.

<u>*0-cells*</u> *are triples* $((k \stackrel{t}{\to} k,\mu,\eta),(k \stackrel{c}{\to} k,\delta,\varepsilon),\psi)$, *consisting of a monad* $(k \stackrel{t}{\to} k,\mu,\eta)$, *a comonad* $(k \stackrel{c}{\to} k,\delta,\varepsilon)$ *and a 2-cell* $tc \stackrel{\psi}{\Rightarrow} ct$ *in* $\mathcal{K}$, *such that*

- $(t \stackrel{(c,\psi)}{\to} t,\delta,\varepsilon)$ *is a comonad in* $\mathrm{Mnd}^\iota(\mathcal{K})$ *and*
- $(c \stackrel{(t,\psi)}{\to} c,\mu,\eta)$ *is a monad in* $\mathrm{Cmd}^\pi(\mathcal{K})$.

<u>*1-cells*</u> $((k \stackrel{t}{\to} k,\mu,\eta),(k \stackrel{c}{\to} k,\delta,\varepsilon),\psi) \stackrel{(W,\alpha,\beta)}{\to} ((k' \stackrel{t'}{\to} k',\mu',\eta'),(k' \stackrel{c'}{\to} k',\delta',\varepsilon'),\psi')$ *are triples, consisting of a 1-cell* $k \stackrel{W}{\to} k'$ *and 2-cells* $t'W \stackrel{\alpha}{\Rightarrow} Wt$ *and* $Wc \stackrel{\beta}{\Rightarrow} c'W$ *in* $\mathcal{K}$, *such that*

- $(t \stackrel{(c,\psi)}{\to} t,\delta,\varepsilon) \stackrel{((W,\alpha),\beta)}{\to} (t' \stackrel{(c',\psi')}{\to} t',\delta',\varepsilon')$ *is a 1-cell in* $\mathrm{Cmd}(\mathrm{Mnd}^\iota(\mathcal{K}))$ *and*
- $(c \stackrel{(t,\psi)}{\to} c,\mu,\eta) \stackrel{((W,\beta),\alpha)}{\to} (c' \stackrel{(t',\psi')}{\to} c',\mu',\eta')$ *is a 1-cell in* $\mathrm{Mnd}(\mathrm{Cmd}^\pi(\mathcal{K}))$.

<u>*2-cells*</u> $(W,\alpha,\beta) \stackrel{\omega}{\Rightarrow} (W',\alpha',\beta')$ *are 2-cells* $W \stackrel{\omega}{\Rightarrow} W'$ *in* $\mathcal{K}$, *such that*

- $((W,\alpha),\beta) \stackrel{\omega}{\Rightarrow} ((W',\alpha'),\beta')$ *is a 2-cell in* $\mathrm{Cmd}(\mathrm{Mnd}^\iota(\mathcal{K}))$ *and*
- $((W,\beta),\alpha) \stackrel{\omega}{\Rightarrow} ((W',\beta'),\alpha')$ *is a 2-cell in* $\mathrm{Mnd}(\mathrm{Cmd}^\pi(\mathcal{K}))$.

Horizontal and vertical compositions are the same as in $\mathcal{K}$.

Proof. In order to see that 0-cells in $\mathrm{Entw}^w(\mathcal{K})$ are precisely the weak entwining structures, note that (1.1) expresses the requirement that $t \stackrel{(c,\psi)}{\to} t$ is a 1-cell in $\mathrm{Mnd}^\iota(\mathcal{K})$ and (1.2) means that $c \stackrel{(t,\psi)}{\to} c$ is a 1-cell in $\mathrm{Cmd}^\pi(\mathcal{K})$. Axiom (1.3) means that $(k,c) \stackrel{\eta}{\Rightarrow} (t,\psi)$ is a 2-cell in $\mathrm{Cmd}^\pi(\mathcal{K})$ and (1.4) holds if and only if $(c,\psi) \stackrel{\varepsilon}{\Rightarrow} (k,t)$ is a 2-cell in $\mathrm{Mnd}^\iota(\mathcal{K})$. If these four conditions hold, then also $(t,\psi)(t,\psi) \stackrel{\mu}{\Rightarrow} (t,\psi)$ is a 2-cell in $\mathrm{Cmd}^\pi(\mathcal{K})$. That is,

$$\begin{aligned} c\varepsilon t * c\psi * c\mu c * \psi tc * t\psi c * tt\delta &\stackrel{(1.1)}{=} c\varepsilon t * c\psi * \psi c * \mu cc * tt\delta = c\varepsilon t * c\psi * \psi c * t\delta * \mu c \\ &\stackrel{(1.2)}{=} c\varepsilon t * \delta t * \psi * \mu c = \psi * \mu c. \end{aligned}$$

Similarly, (1.1-1.4) imply that $(c,\psi) \stackrel{\delta}{\Rightarrow} (c,\psi)(c,\psi)$ is a 2-cell in $\mathrm{Mnd}^\iota(\mathcal{K})$, i.e.

$$\begin{aligned} cc\mu * c\psi t * \psi ct * t\delta t * t\psi * t\eta c &\stackrel{(1.2)}{=} cc\mu * \delta tt * \psi t * t\psi * t\eta c = \delta t * c\mu * \psi t * t\psi * t\eta c \\ &\stackrel{(1.1)}{=} \delta t * \psi * \mu c * t\eta c = \delta t * \psi. \end{aligned}$$

By Theorem 1.1, a triple $(k \stackrel{W}{\to} k', t'W \stackrel{\alpha}{\Rightarrow} Wt, Wc \stackrel{\beta}{\Rightarrow} c'W)$ is a 1-cell $((k \stackrel{t}{\to} k,\mu,\eta),(k \stackrel{c}{\to} k,\delta,\varepsilon),\psi) \to ((k' \stackrel{t'}{\to} k',\mu',\eta'),(k' \stackrel{c'}{\to} k',\delta',\varepsilon'),\psi')$ in $\mathrm{Entw}^w(\mathcal{K})$ if and only if the following equalities hold.

$$\alpha * \mu' W = W\mu * \alpha t * t'\alpha; \tag{1.10}$$

$$\alpha * \eta' W = W\eta; \tag{1.11}$$

$$\delta' W * \beta = c'\beta * \beta c * W\delta; \tag{1.12}$$

$$\varepsilon' W * \beta = W\varepsilon; \tag{1.13}$$

$$c'W\mu * c'\alpha t * \psi' Wt * t'\beta t * t'W\psi * t'W\eta c = \beta t * W\psi * \alpha c \tag{1.14}$$

$$c'W\varepsilon t * c'W\psi * c'\alpha c * \psi' Wc * t'\beta c * t'W\delta = \beta t * W\psi * \alpha c. \tag{1.15}$$

The equality (1.10) is equivalent to saying that $t \stackrel{(W,\alpha)}{\to} t'$ is a 1-cell in $\mathrm{Mnd}^\iota(\mathcal{K})$ and (1.12) is equivalent to $c \stackrel{(W,\beta)}{\to} c'$ being a 1-cell in $\mathrm{Cmd}^\pi(\mathcal{K})$. The equality (1.15) means (after being simplified using (1.13)) that $(t',\psi')(W,\beta) \stackrel{\alpha}{\Rightarrow} (W,\beta)(t,\psi)$ is a 2-cell in $\mathrm{Cmd}^\pi(\mathcal{K})$ and (1.14) means (after being simplified using (1.11)) that $(W,\alpha)(c,\psi) \stackrel{\beta}{\Rightarrow} (c',\psi')(W,\alpha)$ is a 2-cell in $\mathrm{Mnd}^\iota(\mathcal{K})$. Conditions (1.10) and (1.11) mean that $(c \stackrel{(t,\psi)}{\to} c,\mu,\eta) \stackrel{((W,\beta),\alpha)}{\Rightarrow} (c' \stackrel{(t',\psi')}{\to} c',\mu',\eta')$ is a 2-cell in $\mathrm{Mnd}(\mathrm{Cmd}^\pi(\mathcal{K}))$, while (1.12) and (1.13) express that $(t \stackrel{(c,\psi)}{\to} t,\delta,\varepsilon) \stackrel{((W,\alpha),\beta)}{\Rightarrow} (t' \stackrel{(c',\psi')}{\to} t',\delta',\varepsilon')$ is a 2-cell in $\mathrm{Cmd}(\mathrm{Mnd}^\iota(\mathcal{K}))$.

A 2-cell $W \stackrel{\omega}{\Rightarrow} W'$ in $\mathcal{K}$ is a 2-cell $(W,\alpha,\beta) \Rightarrow (W',\alpha',\beta')$ in $\mathrm{Entw}^w(\mathcal{K})$ if and only if

$$\alpha' * t'\omega = \omega t * \alpha \tag{1.16}$$

$$\beta' * \omega c = c'\omega * \beta. \tag{1.17}$$

For any weak entwining structure $((k \stackrel{t}{\to} k,\mu,\eta),(k \stackrel{c}{\to} k,\delta,\varepsilon),\psi)$ in $\mathcal{K}$, the triple $(W = k, \alpha = t, \beta = c)$ satisfies the equalities (1.10-1.15). Hence it is an (identity) 1-cell in $\mathrm{Entw}^w(\mathcal{K})$. The sets of 1-cells and 2-cells in $\mathrm{Cmd}(\mathrm{Mnd}^\iota(\mathcal{K}))$ and $\mathrm{Mnd}(\mathrm{Cmd}^\pi(\mathcal{K}))$ are closed under the horizontal composition in $\mathcal{K}$ by Theorem 1.1. Therefore the

horizontal composite of 1-cells and 2-cells in $\mathrm{Entw}^w(\mathcal{K})$ is a 1-cell and a 2-cell in $\mathrm{Entw}^w(\mathcal{K})$, respectively.

For any 1-cell (W,α,β) in $\mathrm{Entw}^w(\mathcal{K})$, the identity 2-cell $W \overset{W}{\Rightarrow} W$ in $\mathcal{K}$ satisfies (1.16) and (1.17). Hence it is an (identity) 2-cell in $\mathrm{Entw}^w(\mathcal{K})$. Since the sets of 2-cells in $\mathrm{Cmd}(\mathrm{Mnd}^\iota(\mathcal{K}))$ and $\mathrm{Mnd}(\mathrm{Cmd}^\pi(\mathcal{K}))$ are closed under the vertical composition in $\mathcal{K}$ by Theorem 1.1, the vertical composite of 2-cells in $\mathrm{Entw}^w(\mathcal{K})$ is a 2-cell in $\mathrm{Entw}^w(\mathcal{K})$ again.

Associativity and unitality of the horizontal and vertical compositions in $\mathrm{Entw}^w(\mathcal{K})$ and the interchange law follow by the respective properties of $\mathcal{K}$. $\square$

From Theorem 1.2, we immediately deduce the existence of some 2-functors.

Corollary 1.3. *For any 2-category $\mathcal{K}$, the following assertions hold.*

(1) *There is a 2-functor $Y : \mathcal{K} \to \mathrm{Entw}^w(\mathcal{K})$, determined by the maps $k \mapsto (I(k), I_*(k), k)$, $V \to (V,V,V)$ and $\omega \mapsto \omega$ on the 0-, 1-, and 2-cells, respectively.*

(2) *There is a 2-category isomorphism $\Phi : \mathrm{Entw}^w(\mathcal{K}) \cong \mathrm{Entw}^w(\mathcal{K}_*)_*$, determined by the maps $(t,c,\psi) \mapsto (c,t,\psi)$, $(W,\alpha,\beta) \mapsto (W,\beta,\alpha)$ and $\omega \mapsto \omega$ on the 0-, 1-, and 2-cells, respectively. In particular, for any weak entwining structures (t,c,ψ) and (t',c',ψ') in $\mathcal{K}$, there is a category isomorphism $\mathrm{Entw}^w(\mathcal{K})((t,c,\psi),(t',c',\psi')) \cong \mathrm{Entw}^w(\mathcal{K}_*)_*((c,t,\psi),(c',t',\psi'))$, which is 2-natural both in (t,c,ψ) and (t',c',ψ').*

(3) *There is a 2-functor $A : \mathrm{Entw}^w(\mathcal{K}) \to \mathrm{Cmd}(\mathrm{Mnd}^\iota(\mathcal{K}))$, determined by the maps $((t,\mu,\eta),(c,\delta,\varepsilon),\psi) \mapsto (t \overset{(c,\psi)}{\to} t,\delta,\varepsilon)$, $(W,\alpha,\beta) \mapsto ((W,\alpha),\beta)$ and $\omega \mapsto \omega$ on the 0-, 1-, and 2-cells, respectively.*

(4) *There is a 2-functor $B : \mathrm{Entw}^w(\mathcal{K}) \to \mathrm{Mnd}(\mathrm{Cmd}^\pi(\mathcal{K}))$, determined by the maps $((t,\mu,\eta),(c,\delta,\varepsilon),\psi) \mapsto (c \overset{(t,\psi)}{\to} c,\mu,\eta)$, $(W,\alpha,\beta) \mapsto ((W,\beta),\alpha)$ and $\omega \mapsto \omega$ on the 0-, 1-, and 2-cells, respectively.*

In contrast to the case of usual entwining structures, there seems to be no reason to expect that the 2-functors A and B in Corollary 1.3 are isomorphisms.

2. Equivalence of Eilenberg-Moore objects

If a 2-category $\mathcal{K}$ admits Eilenberg-Moore constructions for both monads and comonads and idempotent 2-cells in $\mathcal{K}$ split, then by Theorem 1.1 and Corollary 1.3, there are two pseudo-functors $J_*\mathrm{Cmd}(Q)A$ and $J\mathrm{Mnd}(Q_*)B : \mathrm{Entw}^w(\mathcal{K}) \to \mathcal{K}$. The aim of this section is to prove that both are right biadjoints of Y in Corollary 1.3(1), hence they are pseudo-naturally equivalent. Consequently, for any weak entwining structure (t,c,ψ) in $\mathcal{K}$, the monad $Q_*(c \overset{(t,\psi)}{\to} c)$ and the comonad $Q(t \overset{(c,\psi)}{\to} t)$ in $\mathcal{K}$ possess equivalent Eilenberg-Moore objects.

Recall that any pseudo-functor $Q : \mathcal{A} \to \mathcal{B}$ between 2-categories, induces a pseudo-functor $\mathrm{Cmd}(Q) : \mathrm{Cmd}(\mathcal{A}) \to \mathrm{Cmd}(\mathcal{B})$ with underlying maps as follows. A comonad $(A \overset{c}{\to} A,\delta,\varepsilon)$ in $\mathcal{A}$ is taken to the comonad $Q(A) \overset{Q(c)}{\to} Q(A)$, with comultiplication $Q(c) \overset{Q(\delta)}{\Rightarrow} Q(cc) \overset{\cong}{\Rightarrow} Q(c)Q(c)$ and counit $Q(c) \overset{Q(\varepsilon)}{\Rightarrow} Q(1_A) \overset{\cong}{\Rightarrow} 1_{Q(A)}$. A 1-cell $(A \overset{c}{\to} A,\delta,\varepsilon) \overset{(V,\psi)}{\to} (A' \overset{c'}{\to} A',\delta',\varepsilon')$ in $\mathrm{Cmd}(\mathcal{A})$ is taken to a pair consisting of the 1-cell

$Q(A) \stackrel{Q(V)}{\to} Q(A')$ and the 2-cell $Q(V)Q(c) \stackrel{\cong}{\Rightarrow} Q(Vc) \stackrel{Q(\psi)}{\Rightarrow} Q(c'V) \stackrel{\cong}{\Rightarrow} Q(c')Q(V)$ in $\mathcal{B}$. A 2-cell ω in $\mathrm{Cmd}(\mathcal{A})$ is taken to $Q(\omega)$. $\mathrm{Cmd}(Q)$ is a pseudo-functor with the same coherence isomorphisms as Q.

Proposition 2.1. *Consider a 2-category $\mathcal{K}$ which admits Eilenberg-Moore constructions for monads and in which idempotent 2-cells split. Let l be a 0-cell and $((k \stackrel{t}{\to} k, \mu, \eta), (k \stackrel{c}{\to} k, \delta, \varepsilon), \psi)$ be weak entwining structure in $\mathcal{K}$. The following categories are isomorphic.*

(i) *The Eilenberg-Moore category* $\mathrm{Cmd}(\mathcal{K})(I_*(l), \mathrm{Cmd}(Q)(t \stackrel{(c,\psi)}{\to} t, \delta, \varepsilon))$ *of the comonad* $\mathcal{K}(l, Q(t \stackrel{(c,\psi)}{\to} t)) : \mathcal{K}(l, Q(t)) \to \mathcal{K}(l, Q(t))$;

(ii) *the category* $\mathrm{Entw}^w(\mathcal{K})(Y(l), (t, c, \psi))$.

Moreover, these isomorphisms provide the 1-cell parts of a pseudo-natural isomorphism $\mathrm{Cmd}(\mathcal{K})(I_*(-), \mathrm{Cmd}(Q)A(-)) \cong \mathrm{Entw}^w(\mathcal{K})(Y(-), -)$.

Proof. By (1.10-1.15), the objects in the category $\mathrm{Entw}^w(\mathcal{K})(Y(l), (t, c, \psi))$ are triples $(l \stackrel{W}{\to} k, tW \stackrel{\varrho}{\Rightarrow} W, W \stackrel{\kappa}{\Rightarrow} cW)$, such that $I(l) \stackrel{(W,\varrho)}{\to} t$ is a 1-cell in $\mathrm{Mnd}(\mathcal{K})$, $I_*(l) \stackrel{(W,\kappa)}{\to} c$ is a 1-cell in $\mathrm{Cmd}(\mathcal{K})$ and

$$c\varrho * \psi W * t\kappa = \kappa * \varrho. \tag{2.1}$$

Morphisms $(W, \varrho, \kappa) \to (W', \varrho', \kappa')$ in $\mathrm{Entw}^w(\mathcal{K})(Y(l), (t, c, \psi))$ are 2-cells $W \stackrel{\omega}{\Rightarrow} W'$ in $\mathcal{K}$, such that $(W, \varrho) \stackrel{\omega}{\Rightarrow} (W', \varrho')$ is a 2-cell in $\mathrm{Mnd}(\mathcal{K})$ and $(W, \kappa) \stackrel{\omega}{\Rightarrow} (W', \kappa')$ is a 2-cell in $\mathrm{Cmd}(\mathcal{K})$. We prove that the stated isomorphism is given by

$$\begin{aligned}
\mathrm{Entw}^w(\mathcal{K})(Y(l), (t, c, \psi)) &\to \mathrm{Cmd}(\mathcal{K})(I_*(l), \mathrm{Cmd}(Q)((c, \psi), \delta, \varepsilon)),\\
(W, \varrho, \kappa) \stackrel{\omega}{\to} (W', \varrho', \kappa') &\mapsto \mathrm{Cmd}(Q)((W, \varrho), \kappa) \stackrel{Q(\omega)}{\to} \mathrm{Cmd}(Q)((W', \varrho'), \kappa').
\end{aligned}$$

If applying the convention of choosing trivial splittings of identity 2-cells, as described in Theorem 1.1, then when restricted to the 2-subcategory $\mathrm{Mnd}(\mathcal{K})$ of $\mathrm{Mnd}^\iota(\mathcal{K})$, Q is equal to J. Hence by [8, Theorem 2], there is a category isomorphism

$$\begin{aligned}
&\mathcal{K}(l, Q(t)) \to \mathrm{Mnd}(\mathcal{K})(I(l), t), \quad V \stackrel{\omega}{\to} V' \mapsto (vV, v\epsilon V) \stackrel{v\omega}{\to} (vV', v\epsilon V');\\
&\mathrm{Mnd}(\mathcal{K})(I(l), t) \to \mathcal{K}(l, Q(t)), \quad (W, \varrho) \stackrel{\varphi}{\to} (W', \varrho') \mapsto Q(W, \varrho) \stackrel{Q(\varphi)}{\to} Q(W', \varrho').
\end{aligned} \tag{2.2}$$

We claim that there is a bijection also between 2-cells $(W, \varrho) \stackrel{\kappa}{\Rightarrow} (c, \psi)(W, \varrho)$ in $\mathrm{Mnd}^\iota(\mathcal{K})$, and 2-cells $Q(W, \varrho) \stackrel{\xi}{\Rightarrow} Q(c, \psi)Q(W, \varrho)$ in $\mathcal{K}$, for any 1-cell $I(l) \stackrel{(W,\varrho)}{\to} t$ in $\mathrm{Mnd}(\mathcal{K})$. Indeed, for a 2-cell κ as described, $\xi := \big(Q(W, \varrho) \stackrel{Q(\kappa)}{\Rightarrow} Q((c, \psi)(W, \varrho)) \stackrel{\cong}{\Rightarrow} Q(c, \psi)Q(W, \varrho)\big)$ is a 2-cell in $\mathcal{K}$ as needed. Conversely, for a 2-cell ξ as above, use the chosen splitting $cv \stackrel{\pi}{\Rightarrow} vQ(c, \psi) \stackrel{\iota}{\Rightarrow} cv$ of the idempotent 2-cell (1.9) to construct a 2-cell $\kappa := \iota Q(W, \varrho) * v\xi : W \Rightarrow cW$ in $\mathcal{K}$. It satisfies

$$\begin{aligned}
\kappa * \varrho &= \iota Q(W, \varrho) * v\xi * v\epsilon Q(W, \varrho) = \iota Q(W, \varrho) * v\epsilon Q(c, \psi)Q(W, \varrho) * tv\xi\\
&\stackrel{(1.7)}{=} \iota Q(W, \varrho) * \pi Q(W, \varrho) * c\varrho * \psi W * t\iota Q(W, \varrho) * tv\xi = c\varrho * \psi W * t\kappa,
\end{aligned}$$

where the last equality follows by $\iota f * \pi f * \psi = c\mu * \psi t * \eta ct * \psi \stackrel{(1.1)}{=} \psi * \mu c * \eta tc = \psi$. Hence κ is a 2-cell $(W, \varrho) \Rightarrow (c, \psi)(W, \varrho)$ in $\mathrm{Mnd}^\iota(\mathcal{K})$, as required. In order to see that this correspondence $\kappa \leftrightarrow \xi$ is a bijection, note that by (1.8), $vQ(\iota Q(W, \varrho))$

is equal to the composite of $vQ(c,\psi)Q(W,\varrho) \overset{\iota Q(W,\varrho)}{\Rightarrow} cW$ and the chosen epi 2-cell $cW \Rightarrow vQ((c,\psi)(W,\varrho))$. That is, $Q(\iota Q(W,\varrho))$ is equal to the coherence iso 2-cell $Q(c,\psi)Q(W,\varrho) \overset{\cong}{\Rightarrow} Q((c,\psi)(W,\varrho))$. Hence starting with a 2-cell ξ and iterating both constructions, we re-obtain ξ. In the opposite order, applying both constructions to κ, by (1.8) we get $\iota Q(W,\varrho) * \pi Q(W,\varrho) * \kappa$. This is equal to κ by

$$\iota Q(W,\varrho) * \pi Q(W,\varrho) * \kappa = c\varrho * \psi W * \eta c W * \kappa \overset{(2.1)}{=} \kappa. \tag{2.3}$$

Next we show that $Q(W,\varrho) \overset{Q(\kappa)}{\Rightarrow} Q((c,\psi)(W,\varrho)) \overset{\cong}{\Rightarrow} Q(c,\psi)Q(W,\varrho)$ is a coassociative coaction if and only if $W \overset{\kappa}{\Rightarrow} cW$ is coassociative, and it is counital if and only if κ is counital. Compose the coassociativity condition $Q((c,\psi)\kappa) * Q(\kappa) = Q(\delta(W,\varrho)) * Q(\kappa)$ horizontally by v on the left and compose it vertically by the chosen mono 2-cell $vQ((c,\psi)(c,\psi)(W,\varrho)) \overset{\iota}{\Rightarrow} ccW$ on the left. Applying (1.8), (2.1) and (2.3), the resulting equivalent condition can be written in the form $c\kappa * \kappa = cc\varrho * c\psi W * \psi c W * \eta cc W * \delta W * \kappa$. Since

$$cc\varrho * c\psi W * \psi c W * \eta cc W * \delta W * \kappa \overset{(1.2)}{=} \delta W * c\varrho * \psi W * \eta c W * \kappa \overset{(2.3)}{=} \delta W * \kappa,$$

this proves that the coaction on $Q(W,\varrho)$ is coassociative if and only if κ is so. By (2.2), (1.8) and (2.3), the counitality condition $Q(\varepsilon(W,\varrho)) * Q(\kappa) = Q(W,\varrho)$ is equivalent to $\varepsilon W * \kappa = W$. Thus there is a bijection between the objects of $\mathrm{Cmd}(\mathcal{K})(I_*(l), \mathrm{Cmd}(Q)((c,\psi),\delta,\varepsilon))$ and the objects of $\mathrm{Entw}^w(\mathcal{K})(Y(l),(t,c,\psi))$, as stated.

One can see by similar steps that, for a 2-cell $(W,\varrho) \overset{\omega}{\Rightarrow} (W',\varrho')$ in $\mathrm{Mnd}(\mathcal{K})$, $Q(\omega)$ is a morphism $Q(W,\varrho) \to Q(W',\varrho')$ in $\mathrm{Cmd}(\mathcal{K})(I_*(l), \mathrm{Cmd}(Q)((c,\psi),\delta,\varepsilon))$ if and only if $\kappa' * \omega = c\varrho' * \psi W' * \eta c W' * c\omega * \kappa$. Since

$$c\varrho' * \psi W' * \eta c W' * c\omega * \kappa = c\varrho' * ct\omega * \psi W * t\kappa * \eta W = c\omega * c\varrho * \psi W * t\kappa * \eta W \overset{(2.3)}{=} c\omega * \kappa,$$

we conclude that $Q(\omega)$ is a morphism of $\mathcal{K}(l, Q(c,\psi))$-coalgebras as needed, if and only if ω is a 1-cell $I_*(l) \to c$ in $\mathrm{Cmd}(\mathcal{K})$, i.e. ω is a morphism $(W,\varrho,\kappa) \to (W',\varrho',\kappa')$ in $\mathrm{Entw}^w(\mathcal{K})(Y(l),(t,c,\psi))$. In view of the isomorphism (2.2), this proves the stated isomorphism $\mathrm{Cmd}(\mathcal{K})(I_*(l), \mathrm{Cmd}(Q)((c,\psi),\delta,\varepsilon)) \cong \mathrm{Entw}^w(\mathcal{K})(Y(l),(t,c,\psi))$.

There is a pseudo-natural transformation

$$\mathrm{Entw}^w(Y(-),-) \to \mathrm{Cmd}(\mathcal{K})(\mathrm{Cmd}(Q)AY(-), \mathrm{Cmd}(Q)A(-)), \tag{2.4}$$

with 1-cell parts the functors induced by the pseudo-functor $\mathrm{Cmd}(Q)A$ and 2-cell parts provided by its pseudo-naturality isomorphisms. Recall that AY differs from $\mathrm{Cmd}(I)I_*$ by the inclusion 2-functor $\mathrm{Cmd}(\mathrm{Mnd}(\mathcal{K})) \hookrightarrow \mathrm{Cmd}(\mathrm{Mnd}^\iota(\mathcal{K}))$. Since applying $Q : \mathrm{Mnd}^\iota(\mathcal{K}) \to \mathcal{K}$ after $\mathcal{K} \overset{I}{\to} \mathrm{Mnd}(\mathcal{K}) \hookrightarrow \mathrm{Mnd}^\iota(\mathcal{K})$ we obtain the identity functor $JI = \mathcal{K}$, it follows that $\mathrm{Cmd}(Q)AY(-) = I_*$ as pseudo-functors. Thus (2.4) is, in fact, a pseudo-natural transformation $\mathrm{Entw}^w(Y(-),-) \to \mathrm{Cmd}(\mathcal{K})(I_*(-), \mathrm{Cmd}(Q)A(-))$. Since we already proved that its 1-cells are isomorphisms, it is a pseudo-natural isomorphism, as stated. □

Theorem 2.2. *Let $\mathcal{K}$ be a 2-category which admits Eilenberg-Moore constructions for both monads and comonads and in which idempotent 2-cells split. The following*

diagram of pseudo-functors is commutative, up to a pseudo-natural equivalence.

$$
\begin{tikzcd}
\mathrm{Entw}^w(\mathcal{K}) \arrow[rr, "A"] \arrow[dd, "B"'] & & \mathrm{Cmd}(\mathrm{Mnd}^\iota(\mathcal{K})) \arrow[d, "\mathrm{Cmd}(Q)"] \\
& & \mathrm{Cmd}(\mathcal{K}) \arrow[d, "J_*"] \\
\mathrm{Mnd}(\mathrm{Cmd}^\pi(\mathcal{K})) \arrow[r, "\mathrm{Mnd}(Q_*)"] & \mathrm{Mnd}(\mathcal{K}) \arrow[r, "J"] & \mathcal{K}\,.
\end{tikzcd}
$$

In particular, for any weak entwining structure (t, c, ψ) *in* $\mathcal{K}$*, the monad* $\mathrm{Mnd}(Q_*)(c \stackrel{(t,\psi)}{\to} c, \mu, \eta)$ *and the comonad* $\mathrm{Cmd}(Q)(t \stackrel{(c,\psi)}{\to} t, \delta, \varepsilon)$ *in* $\mathcal{K}$ *possess equivalent Eilenberg-Moore objects.*

Proof. The proof consists of showing that both $J_*\mathrm{Cmd}(Q)A$ and $J\mathrm{Mnd}(Q_*)B$ are right biadjoints of the 2-functor Y in Corollary 1.3(1). Then the claim follows by uniqueness of a biadjoint up to a pseudo-natural equivalence.

On one hand, there is a sequence of pseudo-natural isomorphisms

$$\mathcal{K}(-, J_*\mathrm{Cmd}(Q)A(-)) \cong \mathrm{Cmd}(\mathcal{K})(I_*(-), \mathrm{Cmd}(Q)A(-)) \cong \mathrm{Entw}^w(\mathcal{K})(Y(-), -),$$

where the second isomorphism follows by Proposition 2.1.

On the other hand, applying Proposition 2.1 to the 2-category $\mathcal{K}_*$ (in the third step) and using Corollary 1.3(2) (in the last step), we obtain a sequence of pseudo-natural isomorphisms

$$
\begin{aligned}
\mathcal{K}(-, J\mathrm{Mnd}(Q_*)B(-)) &\cong \mathrm{Mnd}(\mathcal{K})(I(-), \mathrm{Mnd}(Q_*)B(-)) \\
&\cong \mathrm{Cmd}(\mathcal{K}_*)_*(I(-), \mathrm{Mnd}(Q_*)B(-)) \\
&\cong \mathrm{Entw}^w(\mathcal{K}_*)_*(\Phi Y(-), \Phi(-)) \cong \mathrm{Entw}^w(\mathcal{K})(Y(-), -).
\end{aligned}
$$

□

Example 2.3. Consider the 2-subcategory $\mathcal{K}$ of CAT, whose 1-cells are functors induced by bimodules. Explicitly, 0-cells be module categories M_R for algebras R over a fixed commutative ring k. The 1-cells $M_R \to M_{R'}$ be R-R' bimodules V, i.e. functors $(-)\otimes_R V : M_R \to M_{R'}$. The 2-cells $V \Rightarrow W$ be R-R' bimodule maps $\omega : V \to W$, i.e. natural transformations $(-)\otimes_R V \stackrel{(-)\otimes_R \omega}{\Rightarrow} (-)\otimes_R W$.

A weak entwining structure in $\mathcal{K}$ is then a triple $(t := (-)\otimes_R T, c := (-)\otimes_R C, \psi := (-)\otimes_R \Psi)$, where R is a k-algebra, T is an R-ring (i.e. a monad $R \stackrel{T}{\to} R$ in BIM_k), C is an R-coring (i.e. a comonad $R \stackrel{C}{\to} R$ in BIM_k), and $\Psi : C\otimes_R T \to T\otimes_R C$ is an R-bimodule map such that the equalities (1.1-1.4) hold true.

In this particular 2-category $\mathcal{K}$, the idempotent 2-cell (1.9) is given by an idempotent map. Taking its obvious splitting through its range, the associated pseudo-functor $Q : \mathrm{Mnd}^\iota(\mathcal{K}) \to \mathcal{K}$ in Theorem 1.1 becomes a 2-functor. Hence the isomorphisms in Proposition 2.1 become 2-natural, so that the equivalent Eilenberg-Moore objects in Theorem 2.2 become isomorphic.

Under the minor restriction that $R = k$, the monad $\mathrm{Mnd}(Q_*)B((-)\otimes_R T, (-)\otimes_R C, (-)\otimes_R \Psi)$ and the comonad $\mathrm{Cmd}(Q)A((-)\otimes_R T, (-)\otimes_R C, (-)\otimes_R \Psi)$ were described in [5, Section 2]. It was shown in [4, Proposition 2.3] that their Eilenberg-Moore

categories are isomorphic to the category of so-called weak entwining structures. Using the constructions in the current paper, this category of weak entwining structures is nothing but $\mathrm{Entw}^w(\mathcal{K})(Y(k),((-)\otimes_R T,(-)\otimes_R C,(-)\otimes_R \Psi))$.

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Research Institute for Particle and Nuclear Physics, Budapest,
H-1525 Budapest 114, P.O.B. 49, Hungary
E-mail address: G.Bohm@rmki.kfki.hu